

# Automorphisms of Chevalley groups of types $B_2$ and $G_2$ over local rings<sup>1</sup>

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## Abstract.

In the paper we prove that every automorphism of any adjoint Chevalley group of types  $B_2$  or  $G_2$  is standard, i.e., it is a composition of the “inner” automorphism, ring automorphism and central automorphism.

## INTRODUCTION.

An associative ring  $R$  with a unit is called *local*, if it has exactly one maximal ideal (which coincides with the Jacobson radical of  $R$ ). Equivalently, all non-invertible elements of  $R$  form an ideal. In this paper all rings under consideration are commutative.

Let  $G_{ad}$  be a Chevalley-Demazure group scheme associated with an irreducible root system  $\Phi$  of type  $B_2$  or  $G_2$  (see detailed definitions in the next section);  $G_{ad}(R, \Phi)$  be a set of points  $G$  with values in  $R$ ;  $E_{ad}(R, \Phi)$  be the elementary subgroup of  $G_{ad}(R, \Phi)$ , where  $R$  is a local commutative ring with 1. In this paper we describe automorphisms of  $G_{ad}(R, \Phi)$  and  $E_{ad}(R, \Phi)$  (for the root systems  $A_l, D_l$  and  $E_l$  the automorphisms were described in the paper [1]). Suppose that  $R$  is a local ring with  $1/2$  and  $1/3$ . Then every automorphism of  $G_{ad}(R, \Phi)$  ( $E_{ad}(R, \Phi)$ ) is standard (see below definitions of standard automorphisms). These results for Chevalley groups over fields were proved by R. Steinberg [2] for finite case and by J. E. Humphreys [3] for infinite case. K. Suzuki [4] studied automorphisms of Chevalley groups over rings of  $p$ -adic numbers. E. Abe [5] proved this result for Noetherian rings, but the class of all local rings is not completely contained in the class of Noetherian rings, and the proof of [5] can not be extended to the case of arbitrary local rings.

From the other side, automorphisms of classical groups over rings were discussed by many authors. This field of research was open by Schreier and Van der Varden, who described automorphisms of the group  $PSL_n$  ( $n \geq 3$ ) over arbitrary field. Then J. Diedonne [6], L. Hua and I. Reiner [7], O’Meara [8], B. R. McDonald [9], I. Z. Golubchik and A. V. Mikhalev [10] and others studied this problem for groups over more general rings. To prove our theorem we generalize some methods from the paper of V. M. Petechuk [11].

Every Chevalley group under consideration is embedded into the group  $GL_N(R)$  for some  $N \in \mathbb{N}$ . Therefore we can consider Chevalley groups as matrix groups and use linear algebraic group technique: invertible coordinate changes in local rings, uniqueness of a solution of systems of linear equations over local rings with the condition, that determinant of a corresponding matrix is invertible, and so on. As the result we come to the fact that every automorphism of Chevalley group is induced by automorphism of the corresponding matrix ring.

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## 1. DEFINITIONS AND FORMULATIONS OF MAIN THEOREMS.

## 1.1. Root systems.

**Definition 1.** A finite nonempty set  $\Phi \subset \mathbb{R}^l$  of vectors of the space  $\mathbb{R}^l$  is called a *root system*, if it generates  $\mathbb{R}^l$ , does not contain 0 and satisfies the following properties:

- 1)  $\forall \alpha \in \Phi \ (c \cdot \alpha \in \Phi \Leftrightarrow c = \pm 1)$ ;
- 2) if we introduce

$$\langle \alpha, \beta \rangle := \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \quad (\text{reflection coefficient})$$

for  $\alpha, \beta \in \mathbb{R}^l$ , then for any  $\alpha, \beta \in \Phi$  we have  $\langle \alpha, \beta \rangle \in \mathbb{Z}$ ;

- 3) let for  $\alpha \in \mathbb{R}^l$ ,  $w_\alpha$  be a reflection under a hyperplane, orthogonal to the vector  $\alpha$ , i. e.,  $\forall \beta \in \mathbb{R}^l$

$$w_\alpha(\beta) = \beta - \langle \alpha, \beta \rangle \alpha.$$

Then for any  $\alpha, \beta \in \Phi$  we have  $w_\alpha(\beta) \in \Phi$ , i. e., the set  $\Phi$  is invariant under the action of all reflections  $w_\alpha$ ,  $\alpha \in \Phi$ .

If  $\Phi$  is a root system, then its elements are called *roots*.

**Definition 2.** The group  $W$  generated by all reflections  $w_\alpha$ ,  $\alpha \in \Phi$ , is called a *Weil group* of the system  $\Phi$ .

**Definition 3.** If we put in the space  $\mathbb{R}^l$  a hyperplane, that does not contain any roots from  $\Phi$ , then all roots are divided into two disjoint sets of *positive* ( $\Phi^+$ ) and *negative* ( $\Phi^-$ ) roots. A *system of simple roots* is a set  $\Delta = \{\alpha_1, \dots, \alpha_l\} \subset \Phi^+$  such that any positive root  $\beta \in \Phi^+$  is uniquely written in the form  $n_1\alpha_1 + \dots + n_l\alpha_l$ , where  $n_1, \dots, n_l \in \mathbb{Z}^+$ .

For every root system  $\Phi$  there exists a system of simple roots. The number  $l$  is called a *rank* of the system  $\Phi$ .

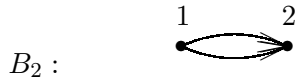
Among all root systems we are interested in *undecomposable systems*, i. e., such systems  $\Phi$  that can not be represented as a union  $\Phi = \Phi_1 \cup \Phi_2$  of two disjoint sets with mutually orthogonal roots.

**Definition 4.** By any root system one can construct the following *Dynkin diagram*. It is a graph that is constructed as follows: its vertices correspond to the simple roots  $\alpha_1, \dots, \alpha_l$ , two vertices with numbers  $i$  and  $j$  are connected, if  $\langle \alpha_i, \alpha_j \rangle \neq 0$ . If  $|\alpha_i| = |\alpha_j|$ , then  $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle$  and the number of edges between vertices  $i$  and  $j$  is equal to  $|\langle \alpha_i, \alpha_j \rangle|$ . If  $|\alpha_i| > |\alpha_j|$ , and  $\langle \alpha_i, \alpha_j \rangle < \langle \alpha_j, \alpha_i \rangle$  and  $|\langle \alpha_i, \alpha_j \rangle| = 1$ . In this case the vertices  $i$  and  $j$  are connected by  $|\langle \alpha_j, \alpha_i \rangle|$  edges and an arrow comes from a long root to a short one.

By a Dynkin diagram we can uniquely define a root system.

All undecomposable root systems up to an isomorphism are divided into 4 infinite (*classical*) series  $A_l$  ( $l \geq 1$ ),  $B_l$  ( $l \geq 2$ ),  $C_l$  ( $l \geq 3$ ) and  $D_l$  ( $l \geq 4$ ) and 5 separate (*exceptional*) cases  $E_6$ ,  $E_7$ ,  $E_8$ ,  $F_4$  and  $G_2$ .

In this paper we are interested in the root systems  $B_2$  and  $G_2$ , with Dynkin diagrams





Here we fix a root system  $\Phi$ , of type  $B_2$  or  $G_2$ , with system of simple roots  $\Delta(B_2) = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2\}$ , or  $\Delta(G_2) = \{\alpha_1 = e_1 - e_2, \alpha_2 = -2e_1 + e_2 + e_3\}$ , positive roots  $\Phi^+(B_2) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2 = e_1, \alpha_1 + 2\alpha_2 = e_1 + e_2\}$ , or  $\Phi^+(G_2) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2 = e_3 - e_1, 2\alpha_1 + \alpha_2 = e_3 - e_2, 3\alpha_1 + \alpha_2 = e_1 + e_3 - 2e_2, 3\alpha_1 + 2\alpha_2 = 2e_3 - e_1 - e_2\}$ , Weil group  $W$ . Recall that in our case in the root system there are roots of two lengths, all roots of the same length are conjugate by the action of the Weil group. More details about root systems and their properties can be found in the books [12], [13].

**1.2. Semisimple Lie algebras.** More details about semisimple Lie algebras can be found in the book [12].

**Definition 5.** Lie algebra  $\mathcal{L}$  over a field  $K$  is a linear space over  $K$ , with an operation of multiplication  $x, y \mapsto [x, y]$ , linear by both variables and satisfying the following conditions:

1) *anticommutativity*:

$$\forall x, y \in \mathcal{L} \quad [x, y] = -[y, x];$$

2) *Jacobi identity*:

$$\forall x, y, z \in \mathcal{L} \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Dimension of Lie algebra is defined by its dimension as a linear space over  $K$ . So Lie algebra is called *finite dimensional*, if the space  $\mathcal{L}$  is finite dimensional.

**Definition 6.** A subspace  $\mathcal{L}'$  of a Lie algebra  $\mathcal{L}$  as a linear space is called a *subalgebra* of  $\mathcal{L}$ , if  $\forall x, y \in \mathcal{L}'$   $[x, y] \in \mathcal{L}'$ . A subalgebra  $\mathcal{L}'$  of  $\mathcal{L}$  is called its *ideal*, if  $\forall x \in \mathcal{L}' \quad \forall y \in \mathcal{L} \quad [x, y] \in \mathcal{L}'$ . A *commutant* of Lie algebras  $\mathcal{L}_1$  and  $\mathcal{L}_2$ , lying in one Lie algebra  $\mathcal{L}$ , is its subalgebra  $\mathcal{L}' = [\mathcal{L}_1, \mathcal{L}_2]$ , generated by all  $[x, y]$  for  $x \in \mathcal{L}_1, y \in \mathcal{L}_2$ .

**Definition 7.** A sequence

$$\mathcal{L}^0 = \mathcal{L}, \mathcal{L}^1 = [\mathcal{L}, \mathcal{L}^0], \mathcal{L}^2 = [\mathcal{L}, \mathcal{L}^1], \dots, \mathcal{L}^{n+1} = [\mathcal{L}, \mathcal{L}^n], \dots$$

is called a *central series* of a Lie algebra  $\mathcal{L}$ . If for some  $n \in \mathbb{N}$  we have  $\mathcal{L}^n = 0$ , then a Lie algebra  $\mathcal{L}$  is called *nilpotent*.

**Definition 8.** A sequence

$$\mathcal{L}^{(0)} = \mathcal{L}, \mathcal{L}^{(1)} = [\mathcal{L}^{(0)}, \mathcal{L}^{(0)}], \mathcal{L}^{(2)} = [\mathcal{L}^{(1)}, \mathcal{L}^{(1)}], \dots, \mathcal{L}^{(n+1)} = [\mathcal{L}^{(n)}, \mathcal{L}^{(n)}], \dots$$

is called a *derivative series* of a Lie algebra  $\mathcal{L}$ . If for some  $n \in \mathbb{N}$  we have  $\mathcal{L}^{(n)} = 0$ , then  $\mathcal{L}$  is called *solvable*.

**Definition 9.** Consider a finitely dimensional Lie algebra  $\mathcal{L}$ . The greatest solvable ideal of Lie algebra, that is a sum of all its solvable ideals, is called a *radical* of Lie algebra. A Lie algebra with zero radical is called *semisimple*. A noncommutative Lie algebra  $\mathcal{L}$  is called *simple*, if it has exactly two ideals: 0 and  $\mathcal{L}$ .

Finitely dimensional semisimple Lie algebra over  $\mathbb{C}$  is a direct sum of simple Lie algebras.

**Definition 10.** A *normalizer* of a subalgebra  $\mathcal{L}'$  in an algebra  $\mathcal{L}$  is a subalgebra

$$N_{\mathcal{L}}(\mathcal{L}') := \{x \in \mathcal{L} \mid \forall y \in \mathcal{L}' \quad [x, y] \in \mathcal{L}'\}.$$

**Definition 11.** A *Cartan subalgebra* of a Lie algebra  $\mathcal{L}$  is its nilpotent self-normalizing subalgebra  $\mathcal{H}$ . For a semisimple Lie algebra it is Abelian and is defined up to an automorphism of the algebra.

**Proposition 1.** ([12], § 8) Let  $\mathcal{L}$  be a semisimple finitely dimensional Lie algebra over  $\mathbb{C}$ ,  $\mathcal{H}$  its Cartan subalgebra. Consider the space  $\mathcal{H}^*$ . Let for  $\alpha \in \mathcal{H}^*$

$$\mathcal{L}_\alpha := \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x \text{ for any } h \in \mathcal{H}\}.$$

In this case  $\mathcal{L}_0 = \mathcal{H}$  and the algebra  $\mathcal{L}$  allows a decomposition  $\mathcal{L} = \mathcal{H} \oplus \sum_{\alpha \neq 0} \mathcal{L}_\alpha$ , and if  $\mathcal{L}_\alpha \neq 0$ , then  $\dim \mathcal{L}_\alpha = 1$ , all such nonzero  $\alpha \in \mathcal{H}^*$  that  $\mathcal{L}_\alpha \neq 0$ , form some root system  $\Phi$ . A root system  $\Phi$  and a semisimple Lie algebra  $\mathcal{L}$  over  $\mathbb{C}$  uniquely define each other.

On a Lie algebra  $\mathcal{L}$  one can introduce a bilinear *Killing form*

$$\kappa(x, y) = \text{tr}(\text{ad } x \text{ ad } y),$$

where  $\text{ad } x \in GL(\mathcal{L})$ ,  $\text{ad } x : z \mapsto [x, z]$  is an *adjoint representation*. For a semisimple Lie algebra a restriction of the Killing form on  $\mathcal{H}$  is non-degenerate, so we can identify the spaces  $\mathcal{H}$  and  $\mathcal{H}^*$ .

**Proposition 2.** ([14], p. 10) There exists a basis  $\{h_1, \dots, h_l\}$  in  $\mathcal{H}$  and for every  $\alpha \in \Phi$  elements  $x_\alpha \in \mathcal{L}_\alpha$  so that

- 1)  $\{h_i; x_\alpha\}$  is a basis in  $\mathcal{L}$ ;
- 2)  $[h_i, h_j] = 0$ ;
- 3)  $[h_i, x_\alpha] = \langle \alpha, \alpha_i \rangle x_\alpha$ ;
- 4)  $[x_\alpha, x_{-\alpha}] = h_\alpha =$  is an integral linear combination of  $h_i$ ;
- 5)  $[x_\alpha, x_\beta] = N_{\alpha\beta} x_{\alpha+\beta}$ ,  $\alpha + \beta \in \Phi$  ( $N_{\alpha\beta} \in \mathbb{Z}$ );
- 6)  $[x_\alpha, x_\beta] = 0$ , if  $\alpha + \beta \neq 0$ ,  $\alpha + \beta \notin \Phi$ .

**Definition 12.** A *representation* of Lie algebra  $\mathcal{L}$  in a linear space  $V$  is a linear mapping  $\pi : \mathcal{L} \rightarrow gl(V)$ , with

$$\forall x, y \in \mathcal{L} \quad \pi([x, y]) = \pi(x)\pi(y) - \pi(y)\pi(x).$$

A representation is called *faithful*, if it has zero kernel.

**1.3. Elementary Chevalley groups.** Introduce now elementary Chevalley groups (see [14]).

Let  $\mathcal{L}$  be a semisimple Lie algebra (over  $\mathbb{C}$ ) with the root system  $\Phi$ ,  $\pi : \mathcal{L} \rightarrow gl(V)$  be its finitely dimensional faithful representation (of dimension  $n$ ). Then we can choose a basis in the space  $V$  (for example, we can take a basis of weight vectors) so that all operators  $\pi(x_\alpha)^k/k!$  for  $k \in \mathbb{N}$  are written as integral (nilpotent) matrices. An integral matrix can be also considered as a matrix over an arbitrary commutative ring with a unit. Let  $R$  be a ring of this type. Consider the  $n \times n$  matrices over  $R$ , the matrices  $\pi(x_\alpha)^k/k!$  for  $\alpha \in \Phi$ ,  $k \in \mathbb{N}$  are included in  $M_n(R)$ .

Now consider automorphisms of the free module  $R^n$  of the form

$$\exp(tx_\alpha) = x_\alpha(t) = 1 + t\pi(x_\alpha) + t^2\pi(x_\alpha)^2/2 + \dots + t^k\pi(x_\alpha)^k/k! + \dots$$

Since all matrices  $\pi(x_\alpha)$  are nilpotent, we have that this series is finite.

**Definition 13.** The subgroup of  $Aut(R^n)$ , generated by all automorphisms  $x_\alpha(t)$ ,  $\alpha \in \Phi$ ,  $t \in R$ , is called an *elementary Chevalley group* (notation:  $E_\pi(\Phi, R)$ ).

**Definition 14.** In elementary Chevalley group we can introduce the following important elements and subgroups:

- $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$ ,  $\alpha \in \Phi$ ,  $t \in R^*$ ;
- $h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1}$ ;
- the subgroup  $N$  is generated by all  $w_\alpha(t)$ ,  $\alpha \in \Phi$ ,  $t \in R^*$ ;

— the subgroup  $H$  is generated by all  $h_\alpha(t)$ ,  $\alpha \in \Phi$ ,  $t \in R^*$ .

It is known that the group  $N$  is a normalizer of  $H$  in elementary Chevalley group, the quotient group  $N/H$  is isomorphic to the Weil group  $W(\Phi)$ .

Elementary Chevalley groups are defined not by representation of the Chevalley groups, but just by its *weight lattice*:

**Definition 15.** If  $V$  is the representation space of the semisimple Lie algebra  $\mathcal{L}$  (with the Cartan subalgebra  $\mathcal{H}$ ), then a functional  $\lambda \in \mathcal{H}^*$  is called the *weight* of its representation, if there exists a nonzero vector  $v \in V$  (which is called the *weight vector*) such that for every  $h \in \mathcal{H}$   $\pi(h)v = \lambda(h)v$ . All weights of a given representation (by addition) generate a lattice (free Abelian group, where every  $\mathbb{Z}$ -basis is also a  $\mathbb{C}$ -basis in  $\mathcal{H}^*$ ), that is called the *weight lattice*  $\Lambda_\pi$ .

An elementary Chevalley group is completely defined by a root system  $\Phi$ , commutative ring  $R$  with a unit and a weight lattice  $\Lambda_\pi$ .

Among all lattices we can mark two: the lattice corresponding to the adjoint representation, it is generated by all roots (the *adjoint lattice*  $\Lambda_{ad}$ ) and the lattice generated by all weights of all representations (the *universal lattice*  $\Lambda_{sc}$ ). For every faithful representation  $\pi$  we have the inclusion  $\Lambda_{ad} \subseteq \Lambda_\pi \subseteq \Lambda_{sc}$ . Respectively, we have the *adjoint* and *universal* elementary Chevalley groups.

**Proposition 3.** ([14], p. 32) *Every elementary Chevalley group satisfies the following conditions:*

$$(R1) \forall \alpha \in \Phi \forall t, u \in R \quad x_\alpha(t)x_\alpha(u) = x_\alpha(t+u);$$

$$(R2) \forall \alpha, \beta \in \Phi \forall t, u \in R \quad \alpha + \beta \neq 0 \Rightarrow$$

$$[x_\alpha(t), x_\beta(u)] = x_\alpha(t)x_\beta(u)x_\alpha(-t)x_\beta(-u) = \prod x_{i\alpha+j\beta}(c_{ij}t^i u^j),$$

where  $i, j$  are integers, multiplication is taken by all roots  $i\alpha + j\beta$ , permuted in some fixed order;  $c_{ij}$  are integer numbers not depending of  $t$  and  $u$ ;

$$(R3) \forall \alpha \in \Phi \quad w_\alpha = w_\alpha(1);$$

$$(R4) \forall \alpha, \beta \in \Phi \forall t \in R^* \quad w_\alpha h_\beta(t) w_\alpha^{-1} = h_{w_\alpha(\beta)}(t);$$

$$(R5) \forall \alpha, \beta \in \Phi \forall t \in R^* \quad w_\alpha x_\beta(t) w_\alpha^{-1} = x_{w_\alpha(\beta)}(ct), \text{ where } c = c(\alpha, \beta) = \pm 1;$$

$$(R6) \forall \alpha, \beta \in \Phi \forall t \in R^* \forall u \in R \quad h_\alpha(t)x_\beta(u)h_\alpha(t)^{-1} = x_\beta(t^{(\beta, \alpha)}u).$$

By  $X_\alpha$  we denote the subgroup generated by all  $x_\alpha(t)$  for  $t \in R$ .

**1.4. Chevalley groups.** Introduce now Chevalley groups (see [14], [15], [16], [17], and the later references in these papers).

**Definition 16.** A subset  $X \subseteq \mathbb{C}^n$  is called an *affine variety*, if  $X$  is a set of common zeros in  $\mathbb{C}^n$  of a finite system of polynomials from  $\mathbb{C}[x_1, \dots, x_n]$ .

**Definition 17.** Topology of an affine  $n$ -space, where a system of closed sets coincides with the system of affine varieties, is called a *Zariski topology*. A variety is called *irreducible*, if it can not be represented as a union of two proper nonempty closed subsets.

**Definition 18.** Let  $G$  be an affine variety with a group structure. If both mappings

$$m : G \times G \rightarrow G,$$

$$m(x, y) = xy,$$

$$i : G \rightarrow G,$$

$$i(x) = x^{-1}$$

are expressed as polynomials of coordinates, then  $G$  is called an *algebraic group*. A *linear algebraic group* is an arbitrary algebraic subgroup in  $M_n(\mathbb{C})$  (with matrix multiplication).

An algebraic group is called *connected*, if it is irreducible as a variety.

Every algebraic group  $G$  contains the unique greatest connected solvable normal subgroup: a *radical*  $R(G)$ . A connected algebraic group with trivial radical is called *semisimple*.

Consider semisimple linear algebraic groups over algebraically closed fields. These are precisely elementary Chevalley groups  $E_\pi(\Phi, K)$  (see [14], § 5).

All these groups are defined in  $SL_n(K)$  as common set of zeros of polynomials of matrix entries  $a_{ij}$  with integer coefficients (for example, in the case of the root system  $C_l$  and the universal representation we have  $n = 2l$  and the polynomials from the condition  $(a_{ij})Q(a_{ji}) - Q = 0$ ). It is clear now that multiplication and inversion are also defined by polynomials with integer coefficients. Therefore, these polynomials can be considered as polynomial over arbitrary commutative ring with a unit. Let some elementary Chevalley group  $E$  over  $\mathbb{C}$  be defined in  $SL_n(\mathbb{C})$  by polynomials  $p_1(a_{ij}), \dots, p_m(a_{ij})$ . For a commutative ring  $R$  with a unit let us consider the group

$$G(R) = \{(a_{ij} \in M_n(R) | \tilde{p}_1(a_{ij}) = 0, \dots, \tilde{p}_m(a_{ij}) = 0\},$$

where  $\tilde{p}_1(\dots), \dots, \tilde{p}_m(\dots)$  are polynomials having the same coefficients as  $p_1(\dots), \dots, p_m(\dots)$ , but considered over  $R$ .

**Definition 19.** A group described above is called a *Chevalley group*  $G_\pi(\Phi, R)$  of type  $\Phi$  over ring  $R$ , for every algebraically closed field  $K$  it coincides with elementary Chevalley group.

**Definition 20.** The subgroup of diagonal (in the standard basis of weight vectors) matrices of the Chevalley group  $G_\pi(\Phi, R)$  is called the *standard maximal torus* of  $G_\pi(\Phi, R)$  and it is denoted by  $T_\pi(\Phi, R)$ . This group is isomorphic to  $Hom(\Lambda_\pi, R^*)$ .

Let us denote by  $h(\chi)$  the elements of the torus  $T_\pi(\Phi, R)$ , corresponding to the homomorphism  $\chi \in Hom(\Lambda(\pi), R^*)$ .

In particular,  $h_\alpha(u) = h(\chi_{\alpha,u})$  ( $u \in R^*$ ,  $\alpha \in \Phi$ ), where

$$\chi_{\alpha,u} : \lambda \mapsto u^{\langle \lambda, \alpha \rangle} \quad (\lambda \in \Lambda_\pi).$$

Note that the condition

$$G_\pi(\Phi, R) = E_\pi(\Phi, R)$$

is not true even for fields, that are not algebraically closed. But if  $G$  is a universal group and the ring  $R$  is *semilocal* (i.e. it contains only finite number of maximal ideals), then we have the condition  $G_{sc}(\Phi, R) = E_{sc}(\Phi, R)$ . [18], [19], [20], [21].

Let us show the difference between Chevalley groups and their elementary subgroups in the case when a ring  $R$  is semilocal and a Chevalley group is not universal. In this case  $G_\pi(\Phi, R) = E_\pi(\Phi, R)T_\pi(\Phi, R)$  (see [18], [19], [21]), and the elements  $h(\chi)$  are connected with elementary generators by the formula

$$(1) \quad h(\chi)x_\beta(\xi)h(\chi)^{-1} = x_\beta(\chi(\beta)\xi).$$

It is known that the group of elementary matrices  $E_2(R) = E_{sc}(A_1, R)$  is not necessary normal in the special linear group  $SL_2(R) = G_{sc}(A_1, R)$  (see [22], [23], [24]).

But if  $\Phi$  is an irreducible root system of the rank  $l \geq 2$ , then  $E(\Phi, R)$  is always normal in  $G(\Phi, R)$ . In the case of semilocal rings from the formula (1) we see that

$$[G(\Phi, R), G(\Phi, R)] \subseteq E(\Phi, R).$$

If the ring  $R$  also contains  $1/2$ , then it is easy to show that

$$[G(\Phi, R), G(\Phi, R)] = [E(\Phi, R), E(\Phi, R)] = E(\Phi, R).$$

### 1.5. Definitions of standard automorphisms and formulations of main theorems.

**Definition 21.** Let us define three types of automorphisms of the Chevalley group  $G_\pi(\Phi, R)$ , that are called *standard*.

**Central automorphisms.** Let  $C_G(R)$  be the center of  $G_\pi(\Phi, R)$  and  $\tau : G_\pi(\Phi, R) \rightarrow C_G(R)$  is a homomorphism of groups. Then the mapping  $x \mapsto \tau(x)x$  from  $G_\pi(\Phi, R)$  onto itself is an automorphism of  $G_\pi(\Phi, R)$ , that is denoted by the letter  $\tau$  and called a *central automorphism* of  $G_\pi(\Phi, R)$ .

Every central automorphism of the Chevalley group  $G_\pi(\Phi, R)$  is identical on its commutant. By our assumptions the elementary subgroup  $E_\pi(\Phi, R)$  is a commutant of  $G_\pi(\Phi, R)$  and  $E_\pi(\Phi, R)$ , therefore on elementary Chevalley groups all central automorphisms are identical.

**Ring automorphisms.** Let  $\rho : R \rightarrow R$  be an automorphism of the ring  $R$ . The mapping  $x \mapsto \rho \circ x$  from  $G_\pi(\Phi, R)$  onto itself is an automorphism of  $G_\pi(\Phi, R)$ , that is denoted by the same letter  $\rho$  and is called the *ring automorphism* of  $G_\pi(\Phi, R)$ . Note that for all  $\alpha \in \Phi$  and  $t \in R$  an element  $x_\alpha(t)$  is mapped into  $x_\alpha(\rho(t))$ .

**“Inner” automorphisms.** Let  $V$  be the space of the representation  $\pi$  of the group  $G_\pi(\Phi, R)$ ,  $g \in GL(V)$  is such a matrix that  $gG_\pi(\Phi, R)g^{-1} = G_\pi(\Phi, R)$ . Then the mapping  $x \mapsto gxg^{-1}$  from  $G_\pi(\Phi, R)$  onto itself is an automorphism of  $G_\pi(\Phi, R)$ , that is denoted by  $i_g$  and is called the *“inner” automorphism* of  $G_\pi(\Phi, R)$ , induced by the element  $g$  of the group  $GL(V)$ .

Similarly we can define three types of automorphisms of the elementary subgroup  $E(R)$ . An automorphism  $\sigma$  of  $G_\pi(\Phi, R)$  (or  $E_\pi(\Phi, R)$ ) is called *standard*, if it is a composition of automorphisms of these three types.

**Theorem 1.** Let  $E_{ad}(\Phi, R)$  be an elementary Chevalley group with an irreducible root system of types  $B_2$  or  $G_2$ ,  $R$  be a commutative local ring with  $1/2$ . Suppose that if  $\Phi = G_2$  then  $1/3 \in R$ . Then every automorphism of  $E_{ad}(\Phi, R)$  is standard.

In our case we have the same theorem for the Chevalley groups  $G_{ad}(\Phi, R)$ :

**Theorem 2.** Let  $G_{ad}(\Phi, R)$  be a Chevalley group with an irreducible root system of types  $B_2$  or  $G_2$ ,  $R$  be a commutative local ring with  $1/2$ . Suppose that if  $\Phi = G_2$  then  $1/3 \in R$ . Then every automorphism of  $G_{ad}(\Phi, R)$  is standard.

Description of nonstandard automorphisms of the groups  $SL_3(R)$ ,  $GL_3(R)$  over local rings with noninvertible 2 can be found in [25].

The next sections are devoted to the proof of the main theorems.

## 2. REPLACING THE INITIAL AUTOMORPHISM WITH THE SPECIAL ONE.

In this section we use some reasonings from [11].

**Definition 22.** By  $GL_n(R, J)$  we denote the group of matrices  $A$  from  $GL_n(R)$  such that  $A - E \in M_n(J)$ ,  $J$  is the radical of  $R$ .

**Proposition 4.** By an arbitrary automorphism  $\varphi$  of elementary Chevalley group  $E_{ad}(\Phi, R)$  one can construct an isomorphism  $\varphi' = i_{g^{-1}}\varphi$ ,  $g \in GL_n(R)$  of the group  $E_{ad}(\Phi, R) \subset GL_n(R)$  onto some subgroup in  $GL_n(R)$ , such that every matrix  $A \in E_{ad}(\Phi, R)$  with elements from the subring of  $R$ , generated by unit, is mapped under the action of this isomorphism  $\varphi'$  to some matrix from the set  $A \cdot GL_n(R, J)$ .

*Proof.* Let  $J$  be the maximal ideal (radical) of  $R$ ,  $k$  the residue field  $R/J$ . Then  $E_J = E(\Phi, R, J)$  is a group generated by all  $x_\alpha(t)$ ,  $\alpha \in \Phi$ ,  $t \in J$ , is the greatest normal proper subgroup in  $E(\Phi, R)$  (see [19]). Therefore,  $E_J$  is invariant under the action of  $\varphi$ .

By this reason

$$\varphi : E(\Phi, R) \rightarrow E(\Phi, R)$$

induces an automorphism

$$\bar{\varphi} : E(\Phi, R)/E_J = E(\Phi, k) \rightarrow E(\Phi, k).$$

The group  $E(\Phi, k)$  is a Chevalley group over field, therefore automorphism  $\bar{\varphi}$  is standard, i. e., it has a form

$$\bar{\varphi} = i_{\bar{g}} \bar{\rho}, \quad \bar{g} \in E_J(E(\Phi, k)) \quad (\text{see [14], § 10}).$$

It is clear that there exists a matrix  $g \in GL_n(R)$  such that its image under factorization  $R$  by  $J$  is  $\bar{g}$ . Note that it is not necessarily  $g \in N(E(\Phi, R))$ .

Consider the mapping  $\varphi' = i_{g^{-1}} \varphi$ . It is an isomorphism from the group  $E_{ad}(\Phi, R) \subset GL_n(R)$  onto some subgroup of  $GL_n(R)$  such that its image under factorization  $R$  by  $J$  is  $\bar{\rho}$ .

Since the automorphism  $\bar{\rho}$  identically acts on matrices with all elements generated by the unit of  $k$ , then every matrix  $A \in E(\Phi, R)$  with elements from the subring of  $R$  generated by unit, is mapped under the action of  $\varphi'$  into some matrix from the set  $A \cdot GL_n(R, J)$ .  $\square$

Let  $a \in E(\Phi, R)$ ,  $a^2 = 1$ . Then the element  $e = \frac{1}{2}(1 + a)$  is an idempotent in the ring  $M_n(R)$ . This idempotent  $e$  defines a decomposition of the free  $R$ -module  $V = R^n$ :

$$V = eV \oplus (1 - e)V = V_0 \oplus V_1$$

(the modules  $V_0, V_1$  are free, since every projective module over local ring is free). Let  $\bar{V} = \bar{V}_0 \oplus \bar{V}_1$  be decomposition of the  $k$ -module  $\bar{V}$  with respect to  $\bar{a}$ , and  $\bar{e} = \frac{1}{2}(1 + \bar{a})$ .

Then we have

**Proposition 5.** *The modules (subspaces)  $\bar{V}_0, \bar{V}_1$  are images of the modules  $V_0, V_1$  under factorization by  $J$ .*

*Proof.* Let us denote the images of  $V_0, V_1$  under factorization by  $J$  by  $\tilde{V}_0, \tilde{V}_1$ , respectively. Since  $V_0 = \{x \in V | ex = x\}$ ,  $V_1 = \{x \in V | ex = 0\}$ , we have  $\bar{e}(\bar{x}) = \frac{1}{2}(1 + \bar{a})(\bar{x}) = \frac{1}{2}(1 + \bar{a}(\bar{x})) = \frac{1}{2}(1 + \overline{a(x)}) = \overline{e(x)}$ . Then  $\tilde{V}_0 \subseteq \bar{V}_0$ ,  $\tilde{V}_1 \subseteq \bar{V}_1$ .

Let  $x = x_0 + x_1$ ,  $x_0 \in V_0$ ,  $x_1 \in V_1$ . Then  $\bar{e}(\bar{x}) = \bar{e}(\bar{x}_0) + \bar{e}(\bar{x}_1) = \bar{x}_0$ . If  $\bar{x} \in \tilde{V}_0$ , then  $\bar{x} = \bar{x}_0$ .  $\square$

Let  $b = \varphi'(a)$ . Then  $b^2 = 1$  and  $b$  is equivalent to  $a$  modulo  $J$ .

**Proposition 6.** *Suppose that  $a, b \in E_\pi(\Phi, R)$ ,  $a^2 = b^2 = 1$ ,  $a$  is a matrix with elements from the subring of  $R$ , generated by the unit,  $b$  and  $a$  are equivalent modulo  $J$ ,  $V = V_0 \oplus V_1$  is a decomposition of  $V$  with respect to  $a$ ,  $V = V'_0 \oplus V'_1$  is a decomposition of  $V$  with respect to  $b$ . Then  $\dim V'_0 = \dim V_0$ ,  $\dim V'_1 = \dim V_1$ .*

*Proof.* We have an  $R$ -basis of the module  $V$   $\{e_1, \dots, e_n\}$  such that  $\{e_1, \dots, e_k\} \subset V_0$ ,  $\{e_{k+1}, \dots, e_n\} \subset V_1$ . It is clear that

$$\overline{ae_i} = \overline{ae_i} = \overline{\left(\sum_{j=1}^n a_{ij}e_j\right)} = \sum_{j=1}^n \overline{a_{ij}}\bar{e}_j.$$

Let  $\bar{V} = \bar{V}_0 \oplus \bar{V}_1$ ,  $\bar{V} = \bar{V}'_0 \oplus \bar{V}'_1$  are decompositions of  $k$ -module (space)  $\bar{V}$  with respect to  $\bar{a}$  and  $\bar{b}$ . It is clear that  $\bar{V}_0 = \bar{V}'_0$ ,  $\bar{V}_1 = \bar{V}'_1$ . Therefore, by Proposition 5 the images of the modules  $V_0$  and  $V'_0$ ,  $V_1$  and  $V'_1$  under factorization by  $J$  coincide. Let us take such  $\{f_1, \dots, f_k\} \subset V'_0$ ,  $\{f_{k+1}, \dots, f_n\} \subset V'_1$



that  $\bar{f}_i = \bar{e}_i$ ,  $i = 1, \dots, n$ . Since the matrix of transformation from  $\{e_1, \dots, e_n\}$  to  $\{f_1, \dots, f_n\}$  is invertible (it is equivalent to the identical matrix modulo  $J$ ) we have that  $\{f_1, \dots, f_n\}$  is a  $R$ -basis in  $V$ . It is clear that  $\{f_1, \dots, f_k\}$  is a  $R$ -basis in  $V'_0$ ,  $\{v_{k+1}, \dots, v_n\}$  is a  $R$ -basis in  $V'_1$ .  $\square$

### 3. IMAGES OF $w_{\alpha_i}$

Consider an adjoint Chevalley group  $E = E(\Phi, R)$  with one of root systems  $B_2$  or  $G_2$ , its adjoint representation in the group  $GL_{10}(R)$  (or  $GL_{14}(R)$ ), in the basis of weight vectors  $v_1 = x_{\alpha_1}, v_{-1} = x_{-\alpha_1}, \dots, v_l = x_{\alpha_l}, v_{-l} = x_{-\alpha_l}, V_1 = h_1, V_2 = h_2$ , corresponding to the Chevalley basis of  $B_2$ , or  $G_2$ .

We suppose that with the help of the automorphism  $\varphi$  we constructed an isomorphism  $\varphi' = i_{g^{-1}}\varphi$ , described in the previous section. Recall that it is an isomorphism from  $E_{ad}(\Phi, R) \subset GL_n(R)$  onto some subgroup in  $GL_n(R)$ , such that its image under factorization  $R$  by  $J$  coincides with a ring automorphism  $\bar{\rho}$ .

Consider matrices  $h_{\alpha_1}(-1), h_{\alpha_2}(-1)$  (see Definition 14) in our basis. They have the form

$$\begin{aligned} h_{\alpha_1}(-1) &= \text{diag}[-1, -1, -1, -1, 1, 1, 1, 1, 1, 1], \\ h_{\alpha_2}(-1) &= E \end{aligned}$$

for  $B_2$  and

$$\begin{aligned} h_{\alpha_1}(-1) &= \text{diag}[1, 1, -1, -1, -1, -1, -1, -1, -1, -1, 1, 1, 1, 1], \\ h_{\alpha_2}(-1) &= \text{diag}[-1, -1, 1, 1, -1, -1, 1, 1, -1, -1, -1, -1, 1, 1] \end{aligned}$$

for  $G_2$ .

According to Proposition 6 we see that every matrix  $h_i = \varphi'(h_{\alpha_i}(-1))$  in some basis is diagonal with  $\pm 1$  on diagonal, the number of 1 and  $-1$  coincides with this number for  $h_{\alpha_i}(-1)$ . Since  $h_1$  and  $h_2$  commute, there exists a basis, where  $h_1$  and  $h_2$  have the same form as  $h_{\alpha_1}(-1)$  and  $h_{\alpha_2}(-1)$ . Suppose that we come to this basis with the help of the matrix  $g_1$ . It is clear that  $g_1 \in GL_n(R, J)$ . Consider the mapping  $\varphi_1 = i_{g_1}^{-1}\varphi'$ . It is also an isomorphism of the group  $E$  onto some subgroup of  $GL_n(R)$  such that its image under factorization  $R$  by  $J$  is  $\bar{\rho}$ , and  $\varphi_1(h_{\alpha_i}(-1)) = h_{\alpha_i}(-1)$  for  $i = 1, 2$ .

Let us consider the isomorphism  $\varphi_1$ .

**Remark 1.** Every element  $w_i = w_{\alpha_i}(1)$  maps (under conjugation) diagonal matrices into diagonal matrices, i. e., every image of  $w_i$  has block-monomial form. In particular, it can be written as a block-monomial matrix, where the first block is  $8 \times 8$  for  $B_2$  and  $12 \times 12$  for  $G_2$ , and the second block is  $2 \times 2$ .

After conjugation with the matrices from  $GL_n(R, J)$  we come from weight basis to some other basis of  $V$ . Consider the first vector of this new basis, denote it by  $e$ . The Weil group  $W$  transitively acts on the roots of the same length, therefore for every root  $\alpha_i$  of the same length as the first one there exists such  $w^{(\alpha_i)} \in W$  that  $w^{(\alpha_i)}\alpha_1 = \alpha_i$ . Similarly, all roots of the second length are also conjugate up to the action of  $W$ . Let  $\alpha_k$  be the first root of the length not equal to the length of  $\alpha_1$ , and let  $f$  be a  $k$ -th basis vector after the last basis change. If  $\alpha_j$  is a root conjugate to  $\alpha_k$ , then denote by  $w_{(\alpha_j)}$  the element of  $W$  such that  $w_{(\alpha_j)}\alpha_k = \alpha_j$ . Consider now the basis  $e_1, \dots, e_{2m+2}$ , where  $e_1 = e$ ,  $e_k = f$ , and for  $1 < i \leq 2m$  either  $e_i = \varphi_1(w^{(\alpha_i)})e$ , or  $e_i = \varphi_1(w_{(\alpha_i)})f$  depending of the length of  $\alpha_k$ ; for  $2m < i \leq 2m+2$   $e_i$  is not changed. It is clear that the matrix of this basis change is equivalent to 1 modulo  $J$ . Therefore the obtained set of vectors is a basis.

Clear that the matrix  $\varphi_1(w_i)$  ( $i = 1, 2$ ) on the basis part  $\{e_1, \dots, e_{12}\}$  coincides with the matrix  $w_i$  in the initial basis of weight vectors. Since  $h_i(-1)$  are squares of  $w_i$ , their images are not changed in the new basis.

Moreover, we know (Remark 1), that  $\varphi_1(w_i)$  is block-diagonal up to the first 8 (12) and last two elements. Therefore, the last basis part consisting of two elements can be changed independently.

Let us denote matrices  $w_i$  and  $\varphi_1(w_i)$  on this part of basis by  $\tilde{w}_i$  and  $\widetilde{\varphi_1(w_i)}$  respectively, and 2-dimensional module, generated by  $e_{2m+1}$  and  $e_{2m+2}$ , by  $\tilde{V}$ .

**Lemma 1.** *For the root systems  $B_2$  and  $G_2$  there exists such a basis that  $\widetilde{\varphi_1(w_1)}$  and  $\widetilde{\varphi_1(w_2)}$  in this basis have the same form as  $\tilde{w}_1$  and  $\tilde{w}_2$ , i. e. are equal to*

$$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

for the case  $B_2$  and

$$\begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

for the case  $G_2$ .

*Proof.* Since  $\tilde{w}_1$  is an involution and  $\tilde{V}_1^1$  has dimension 1, then there exists a basis  $\{e_1, e_2\}$ , where  $\varphi_1(w_1)$  has the form  $\text{diag}[-1, 1]$ . In the basis  $\{e_1, e_2 - e_1\}$  the matrix  $\varphi_1(w_1)$  has the obtained form for  $B_2$ , and in the basis  $\{e_1, e_2 - 3/2e_1\}$  it has the obtained form for  $G_2$ .

Let the matrix  $\widetilde{\varphi_1(w_2)}$  in this basis be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let us make the basis change with the matrix

$$\begin{pmatrix} 1 & (1-a)/c \\ 0 & 1 + \frac{2(1-a)}{c} \end{pmatrix}.$$

Under this basis change the matrix  $\widetilde{\varphi_1(w_1)}$  is not moved, and the matrix  $\widetilde{\varphi_1(w_2)}$  has the form

$$\begin{pmatrix} 1 & b' \\ c' & d' \end{pmatrix}.$$

Since this matrix is an involution, we have  $c'(1+d') = 0$ ,  $1+b'c' = 1$ . Therefore,  $d' = -1$ ,  $b' = 0$ . Now let us consider the cases  $B_2$  and  $G_2$  separately.

For the case  $B_2$  we can use the condition

$$\left( \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c' & -1 \end{pmatrix} \right)^2 = \left( \begin{pmatrix} 1 & 0 \\ c' & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \right)^2.$$

This condition gives (its second line and first row)  $2c'(c'-2) = 0$ , so  $c' = 2$ .

For the case  $G_2$

$$\left( \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c' & -1 \end{pmatrix} \right)^3 = \left( \begin{pmatrix} 1 & 0 \\ c' & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \right)^3,$$

therefore  $3c'(3c'-1)(c'-1) = 0$ . Since  $3 \in R^*$ ,  $c' \equiv 1 \pmod{J}$ ,  $3c'-1 \equiv 2 \pmod{J}$ , we have  $c' = 1$ .  $\square$

Thus, now we can move to an isomorphism  $\varphi_2$ , that is obtained from  $\varphi_1$  by some basis change with the help of a matrix from  $GL_n(R, J)$ . It has all described above properties of  $\varphi_1$ , but also additionally  $\varphi_2(w_i) = w_i$  for all  $i = 1, \dots, l$ .

Suppose that we now have namely isomorphism  $\varphi_2$  of this form.

We need to consider separately the cases  $B_2$  and  $G_2$ . Having an isomorphism  $\varphi_2$ , replacing all elements of the Weil group, we want with one more basis change move to the new isomorphism  $\varphi_3$ , that has all properties of  $\varphi_2$ , but also does not move all elements  $x_{\alpha_i}(1)$ ,  $\alpha_i \in \Phi$ .

4. IMAGES OF  $x_{\alpha_i}(1)$  IN THE CASE  $B_2$ .

In the case  $B_2$  we have roots (ordered as follows):  $e_1, -e_1, e_2, -e_2, e_1 + e_2, -e_1 - e_2, e_1 - e_2, e_2 - e_1$ . Therefore, the adjoint representation has dimension 10. In this representation

$$w_{e_1 - e_2} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad w_{e_1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \end{pmatrix},$$

$$w_{e_1 + e_2} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad w_{e_2} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \end{pmatrix},$$

$$x_{e_2}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is clear that any conditions that hold for elements of the Chevalley group, hold also and for their images under the isomorphism  $\varphi_2$ . We will use this fact.

Using the condition  $w_{e_1} \cdot x_{e_2}(1) = x_{e_2}(1) \cdot w_{e_1}$ , we obtain

$$x_{e_2} = \varphi_2(x_{e_2}(1)) =$$

$$= \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & a_{1,9} & a_{1,10} \\ a_{1,2} & a_{1,1} & a_{1,3} & a_{1,4} & -a_{1,8} & -a_{1,7} & -a_{1,6} & -a_{1,5} & -a_{1,9} - 2a_{1,10} & a_{1,10} \\ a_{3,1} & a_{3,1} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & -a_{3,6} & -a_{3,5} & a_{3,9} & -a_{3,9} \\ a_{4,1} & a_{4,1} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & -a_{4,6} & -a_{4,5} & a_{4,9} & -a_{4,9} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} & a_{5,7} & a_{5,8} & a_{5,9} & a_{5,10} \\ a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} & a_{6,7} & a_{6,8} & a_{6,9} & a_{6,10} \\ -a_{6,2} & -a_{6,1} & -a_{6,3} & -a_{6,4} & a_{6,8} & a_{6,7} & a_{6,6} & a_{6,5} & a_{6,9} + 2a_{6,10} & -a_{6,10} \\ -a_{5,2} & -a_{5,1} & -a_{5,3} & -a_{5,4} & a_{5,8} & a_{5,7} & a_{5,6} & a_{5,5} & a_{5,9} + 2a_{5,10} & -a_{5,10} \\ a_{9,1} & -a_{9,1} & 0 & 0 & a_{9,5} & a_{9,6} & a_{9,6} & a_{9,5} & a_{9,9} & 0 \\ a_{10,1} & a_{10,1} - 2a_{9,1} & a_{10,3} & a_{10,4} & a_{10,5} & a_{10,6} & 2a_{9,6} - a_{10,6} & 2a_{9,5} - a_{10,5} & a_{10,9} & a_{9,9} - a_{10,9} \end{pmatrix}$$

(it can be checked by direct calculations).

Making basis change with the block-diagonal matrix, that is identical on the basis part

$$\{v_{e_1+e_2}, v_{-e_1-e_2}, v_{e_1-e_2}, v_{e_2-e_1}, V_1, V_2\}$$

and has the form

$$\begin{pmatrix} 2a_{1,7}/(a_{1,7}^2 + a_{1,8}^2) & -2a_{1,8}/(a_{1,7}^2 + a_{1,8}^2) & 0 & 0 \\ -2a_{1,8}/(a_{1,7}^2 + a_{1,8}^2) & 2a_{1,7}/(a_{1,7}^2 + a_{1,8}^2) & 0 & 0 \\ 0 & 0 & 2a_{1,7}/(a_{1,7}^2 + a_{1,8}^2) & -2a_{1,8}/(a_{1,7}^2 + a_{1,8}^2) \\ 0 & 0 & -2a_{1,8}/(a_{1,7}^2 + a_{1,8}^2) & 2a_{1,7}/(a_{1,7}^2 + a_{1,8}^2) \end{pmatrix}$$

on the part  $\{v_{e_1}, v_{-e_1}, v_{e_2}, v_{-e_2}\}$ , we do not move the elements  $w_i$ , but now  $a_{1,7}$  is equal to 2, and  $a_{1,8}$  is equal to zero 0. At the same time this basis change is equivalent to identical one modulo  $J$ .

Similarly, basis change with the help of block-diagonal matrix, that is identical on the basis part  $\{v_{e_1}, v_{-e_1}, v_{e_2}, v_{-e_2}, V_1, V_2\}$  and having the form

$$\begin{pmatrix} 2a_{3,9}/(a_{3,9}^2 + a_{4,9}^2) & -2a_{4,9}/(a_{3,9}^2 + a_{4,9}^2) & 0 & 0 \\ -2a_{4,9}/(a_{3,9}^2 + a_{4,9}^2) & 2a_{3,9}/(a_{3,9}^2 + a_{4,9}^2) & 0 & 0 \\ 0 & 0 & 2a_{3,9}/(a_{3,9}^2 + a_{4,9}^2) & -2a_{4,9}/(a_{3,9}^2 + a_{4,9}^2) \\ 0 & 0 & -2a_{4,9}/(a_{3,9}^2 + a_{4,9}^2) & 2a_{3,9}/(a_{3,9}^2 + a_{4,9}^2) \end{pmatrix}$$

on the part  $\{v_{e_1+e_2}, v_{-e_1-e_2}, v_{e_1-e_2}, v_{e_2-e_1}\}$ , also does not move the elements  $w_i$ ,  $a_{1,7}$ ,  $a_{1,8}$ , but now  $a_{3,9}$  is equal to 2,  $a_{4,9}$  equal to 0. Also this change is identical modulo  $J$ .

We suppose that after these two basis changes we move to an isomorphism  $\varphi_3$ .

Now we consider the matrix  $x_{e_1+e_2}$ , the image of  $x_{e_1+e_2}(1)$ . It commutes with  $h_{e_1-e_2}(-1)$  and  $w_{e_1-e_2}$ , therefore it is equal to

$$\begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{1,3} & -b_{1,4} & b_{1,1} & b_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{2,3} & -b_{2,4} & b_{2,1} & b_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{5,5} & b_{5,6} & b_{5,7} & -b_{5,7} & 0 & b_{5,10} \\ 0 & 0 & 0 & 0 & b_{6,5} & b_{6,6} & b_{6,7} & -b_{6,7} & 0 & b_{6,10} \\ 0 & 0 & 0 & 0 & b_{7,5} & b_{7,6} & b_{7,7} & b_{7,8} & b_{7,9} & b_{7,10} \\ 0 & 0 & 0 & 0 & -b_{7,5} & -b_{7,6} & b_{7,8} & b_{7,7} & b_{7,9} & -b_{7,9} - b_{7,10} \\ 0 & 0 & 0 & 0 & b_{9,5} & b_{9,6} & b_{9,7} & b_{9,8} & b_{9,9} & b_{9,10} \\ 0 & 0 & 0 & 0 & 2b_{9,5} & 2b_{9,6} & b_{9,7} - b_{9,8} & b_{9,8} - b_{9,7} & 0 & b_{9,9} + 2b_{9,10} \end{pmatrix}.$$

We will use the following list of conditions:

$$\text{Con1} := (x_{e_2} h_{e_1+e_2}(-1) x_{e_2} h_{e_1+e_2}(-1) = 1),$$

$$\text{Con2} := (x_{e_1+e_2} x_{e_1-e_2} = x_{e_1-e_2} x_{e_1+e_2}) := (x_{e_1+e_2} w_{e_2} x_{e_1+e_2} w_{e_2}^{-1} = w_{e_2} x_{e_1+e_2} w_{e_2}^{-1} x_{e_1+e_2}),$$

$$\text{Con3} := (x_{e_1+e_2} x_{e_2} = x_{e_2} x_{e_1+e_2}),$$

$$\text{Con4} := (x_{e_1+e_2}^2 x_{e_2} x_{e_1} = x_{e_1} x_{e_2}),$$

$$\text{Con5} := (x_{e_1-e_2} w_{e_1-e_2} x_{e_1-e_2} w_{e_1-e_2}^{-1} z_{e_1-e_2} = w_{e_1-e_2}).$$

Let us denote  $y_1 = a_{1,1} - 1$ ,  $y_{1,2} = a_{1,2}$ ,  $y_3 = a_{1,3}$ ,  $y_4 = a_{1,4}$ ,  $y_5 = a_{1,5}$ ,  $y_6 = a_{1,6}$ ,  $y_7 = a_{1,9}$ ,  $y_8 = a_{1,10}$ ,  $y_9 = a_{3,1}$ ,  $y_{10} = a_{3,3} - 1$ ,  $y_{11} = a_{3,4} + 1$ ,  $y_{12} = a_{3,5}$ ,  $y_{13} = a_{3,6}$ ,  $y_{14} = a_{4,1}$ ,  $y_{15} = a_{4,3}$ ,  $y_{16} = a_{4,4} - 1$ ,  $y_{17} = a_{4,5}$ ,  $y_{18} = a_{4,6}$ ,  $y_{19} = a_{5,1} - 1$ ,  $y_{20} = a_{5,2}$ ,  $y_{21} = a_{5,3}$ ,  $y_{22} = a_{5,4}$ ,  $y_{23} = a_{5,5} - 1$ ,  $y_{24} = a_{5,6}$ ,  $y_{25} = a_{5,7} - 1$ ,  $y_{26} = a_{5,8}$ ,  $y_{27} = a_{5,9}$ ,  $y_{28} = a_{5,10}$ ,  $y_{29} = a_{6,1}$ ,  $y_{30} = a_{6,2}$ ,  $y_{31} = a_{6,3}$ ,  $y_{32} = a_{6,4}$ ,  $y_{33} = a_{6,5}$ ,  $y_{34} = a_{6,6} - 1$ ,  $y_{35} = a_{6,7}$ ,  $y_{36} = a_{6,8}$ ,  $y_{37} = a_{6,9}$ ,  $y_{38} = a_{6,10}$ ,  $y_{39} = a_{9,1}$ ,  $y_{40} = a_{9,5}$ ,  $y_{41} = a_{9,6}$ ,  $y_{42} = a_{9,9} - 1$ ,  $y_{43} = a_{10,1}$ ,  $y_{44} = a_{10,3}$ ,  $y_{45} = a_{10,4} - 1$ ,  $y_{46} = a_{10,5}$ ,  $y_{47} = a_{10,6}$ ,  $y_{48} = a_{10,9}$ ,  $y_{49} = b_{1,1} - 1$ ,  $y_{50} = b_{1,2}$ ,  $y_{51} = b_{1,3}$ ,  $y_{52} = b_{1,4} + 1$ ,  $y_{53} = b_{2,1}$ ,  $y_{54} = b_{2,2} - 1$ ,  $y_{55} = b_{2,3}$ ,  $y_{56} = b_{2,4}$ ,  $y_{57} = b_{5,5} - 1$ ,  $y_{58} = b_{5,6} + 1$ ,  $y_{59} = b_{5,7}$ ,  $y_{60} = b_{5,10} + 1$ ,  $y_{61} = b_{6,5}$ ,  $y_{62} = b_{6,6} - 1$ ,  $y_{63} = b_{6,7}$ ,  $y_{64} = b_{6,10}$ ,  $y_{65} = b_{7,5}$ ,  $y_{66} = b_{7,6}$ ,  $y_{67} = b_{7,7} - 1$ ,  $y_{68} = b_{7,8}$ ,  $y_{69} = b_{7,9}$ ,  $y_{70} = b_{7,10}$ ,  $y_{71} = b_{9,5}$ ,  $y_{72} = b_{9,6} - 1$ ,  $y_{73} = b_{9,7}$ ,  $y_{74} = b_{9,8}$ ,  $y_{75} = b_{9,9} - 1$ ,  $y_{76} = b_{9,10}$ . All these  $y_i$  are from  $J$ . From conditions 1–5 we can choose 76 equalities from the following positions (these equations are linear up to  $y_i$ ):

condition *Con1*: positions (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (1, 9), (1, 10), (3, 1), (3, 3), (3, 4), (3, 5), (3, 6), (3, 10), (4, 1), (4, 3), (4, 4), (5, 1), (5, 2), (5, 4), (5, 6), (5, 7), (5, 8), (5, 9), (6, 6), (9, 1), (9, 4) (29 equalities);

condition *Con2*: positions (1, 3), (2, 3), (3, 3), (5, 5), (5, 6), (5, 7), (5, 8), (5, 9), (5, 10), (6, 8), (9, 5), (9, 6), (9, 9), (10, 5), (10, 8) (15 equalities);

condition *Con3*: positions (1, 1), (1, 2), (1, 4), (1, 6), (1, 7), (2, 5), (2, 6), (2, 9), (2, 10), (3, 1), (3, 2), (3, 4), (3, 6), (3, 7), (3, 10), (5, 1), (5, 2), (5, 4), (5, 5), (5, 6), (5, 7), (5, 9), (8, 2), (8, 4) (24 equalities);

condition *Con4*: positions (5, 6), (5, 8), (5, 10), (6, 6), (7, 6), (10, 6) (6 equalities);

condition *Con5*: positions (2, 4), (6, 7) (2 equalities).

If we write the matrix of this linear system modulo  $J$ , we obtain a matrix  $76 \times 76$  with entries  $0, \pm 1, \pm 2$  and determinant  $2^{36}$  (it is checked by direct calculations). Therefore, its determinant is invertible in  $R$ . Consequently, our system of equations has a unique solution  $y_1 = \dots = y_{76} = 0$ . Thus,  $\varphi_3(x_{\alpha_i}(1)) = x_{\alpha_i}(1)$ .

As for elements  $c_t = \varphi_2(h_{e_1+e_2}(t))$ , since  $c_t$  commutes with  $h_{e_1+e_2}(-1)$  and  $w_{e_1-e_2}$ , we have

$$c_t = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_{1,3} & -c_{1,4} & c_{1,1} & c_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_{2,3} & -c_{2,4} & c_{2,1} & c_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{5,5} & c_{5,6} & c_{5,7} & -c_{5,7} & 0 & c_{5,10} \\ 0 & 0 & 0 & 0 & c_{6,5} & c_{6,6} & c_{6,7} & -c_{6,7} & 0 & c_{6,10} \\ 0 & 0 & 0 & 0 & c_{7,5} & c_{7,6} & c_{7,7} & c_{7,8} & c_{7,9} & c_{7,10} \\ 0 & 0 & 0 & 0 & -c_{7,5} & -c_{7,6} & c_{7,8} & c_{7,7} & c_{7,9} & -c_{7,9} - c_{7,10} \\ 0 & 0 & 0 & 0 & c_{9,5} & c_{9,6} & c_{9,7} & c_{9,8} & c_{9,9} & c_{9,10} \\ 0 & 0 & 0 & 0 & 2c_{9,5} & 2c_{9,6} & c_{9,7} - c_{9,8} & c_{9,8} - c_{9,7} & 0 & c_{9,9} + 2c_{9,10} \end{pmatrix}.$$

From  $c_t x_{e_1-e_2} = x_{e_1-e_2} c_t$  we obtain  $c_{1,2} = c_{1,3} = c_{2,1} = c_{2,4} = c_{5,7} = c_{6,7} = c_{7,5} = c_{7,6} = c_{7,8} = c_{7,9} = c_{9,7} = c_{9,8} = c_{7,10} = 0$ ,  $c_{9,9} = c_{7,7}$ .



$$x_{\alpha_1}(1) = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$x_{\alpha_2}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$h_{\alpha_2}(2) = \text{diag}[2, 1/2, 1/4, 4, 1/2, 2, 1, 1, 2, 1/2, 1/2, 2, 1, 1].$$

The fact that  $x_1 = \varphi_2(x_{\alpha_1}(1))$  commutes with  $h_{\alpha_1}(-1)$  and with  $w_{3\alpha_1+2\alpha_2} = w_2 w_1 w_2^{-1} w_2^{-1}$ , gives

$$x_1 = \begin{pmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{11} & -a_{11} & a_{13} & -3/2a_{13} \\ b_1 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{11} & -b_{11} & b_{13} & -3/2b_{13} \\ 0 & 0 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 & d_9 & d_{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & -f_{10} & -f_9 & -f_8 & -f_7 & f_6 & f_5 & f_4 & f_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -e_{10} & -e_9 & -e_8 & -e_7 & e_6 & e_5 & e_4 & e_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -d_{10} & -d_9 & -d_8 & -d_7 & d_6 & d_5 & d_4 & d_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c_{10} & -c_9 & -c_8 & -c_7 & c_6 & c_5 & c_4 & c_3 & 0 & 0 & 0 & 0 \\ g_1 & g_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{11} & g_{12} & g_{13} & g_{14} \\ -g_1 & -g_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{11} & g_{12} & -g_{13} & g_{14} + 3g_{13} \\ h_1 & h_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{11} & -h_{11} + 3i_{11} & h_{13} & 3/2(i_{14} - h_{13}) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i_{11} & i_{11} & 0 & i_{14} \end{pmatrix}.$$

Similarly, since  $x_2 = \varphi_2(x_{\alpha_2}(1))$  commutes with  $h_{\alpha_2}(-1)$  and  $w_{2\alpha_1+\alpha_2} = w_1 w_2 w_1 w_2^{-1} w_1^{-1}$ , we have

$$x_2 = \begin{pmatrix} j_1 & j_2 & 0 & 0 & j_5 & j_6 & 0 & 0 & j_9 & j_{10} & j_{11} & j_{12} & 0 & 0 \\ k_1 & k_1 & 0 & 0 & k_5 & k_6 & 0 & 0 & k_9 & k_{10} & k_{11} & k_{12} & 0 & 0 \\ 0 & 0 & l_3 & l_4 & 0 & 0 & l_7 & -l_7 & 0 & 0 & 0 & 0 & l_{13} & -2l_{13} \\ 0 & 0 & m_3 & m_4 & 0 & 0 & m_7 & -m_7 & 0 & 0 & 0 & 0 & m_{13} & -2m_{13} \\ -k_6 & -k_5 & 0 & 0 & k_2 & k_1 & 0 & 0 & -k_{12} & -k_{11} & k_{10} & k_9 & 0 & 0 \\ -j_6 & -j_5 & 0 & 0 & j_2 & j_1 & 0 & 0 & -j_{12} & -j_{11} & j_{10} & j_9 & 0 & 0 \\ 0 & 0 & n_3 & n_4 & 0 & 0 & n_7 & n_8 & 0 & 0 & 0 & 0 & n_{13} & n_{14} \\ 0 & 0 & -n_3 & -n_4 & 0 & 0 & n_8 & n_7 & 0 & 0 & 0 & 0 & n_{13} + n_{14} & -n_{14} \\ p_1 & p_2 & 0 & 0 & p_5 & p_6 & 0 & 0 & p_9 & p_{10} & p_{11} & p_{12} & 0 & 0 \\ q_1 & q_2 & 0 & 0 & q_5 & q_6 & 0 & 0 & q_9 & q_{10} & q_{11} & 0 & 0 & 0 \\ -q_6 & -q_5 & 0 & 0 & q_2 & q_1 & 0 & 0 & 0 & -q_{11} & q_{10} & q_9 & 0 & 0 \\ -p_6 & -p_5 & 0 & 0 & p_2 & p_1 & 0 & 0 & -p_{12} & -p_{11} & p_{10} & p_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_7 + s_8 & s_7 + s_8 & 0 & 0 & 0 & 2s_{13} + s_{14} & 0 \\ 0 & 0 & s_3 & s_4 & 0 & 0 & s_7 & s_8 & 0 & 0 & 0 & 0 & s_{13} & s_{14} \end{pmatrix}.$$

Finally, since  $h_{\alpha_2}(2)$  commutes with  $h_{\alpha_1}(-1)$ ,  $h_{\alpha_2}(-1)$ ,  $w_{2\alpha_1+\alpha_2}$ , and also from the equality  $Con6 := (w_2 h_{\alpha_2}(2) w_2^{-1} = h_{\alpha_2}(2)^{-1})$ , we obtain

$$d_2 = \varphi_2(h_{\alpha_2}(2)) = \begin{pmatrix} t_1 & t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{11} & t_{12} & 0 & 0 \\ u_1 & u_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_{11} & u_{12} & 0 & 0 \\ 0 & 0 & v_3 & v_4 & 0 & 0 & v_7 & -v_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w_3 & w_4 & 0 & 0 & w_7 & -w_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_2 & u_1 & 0 & 0 & -u_{12} & -u_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_2 & t_1 & 0 & 0 & -t_{12} & -t_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & x_4 & 0 & 0 & x_7 & x_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_3 & -x_4 & 0 & 0 & x_8 & x_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_5 & y_6 & 0 & 0 & y_9 & y_{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_5 & z_6 & 0 & 0 & z_9 & z_{10} & 0 & 0 & 0 & 0 & 0 \\ -z_6 & -z_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_{10} & z_9 & 0 & 0 \\ -y_6 & -y_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{10} & y_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Pos. (14, 12) of  $Con7 := (x_1^2 d_2 = d_2 x_1)$  gives

$$i_{11} \cdot \alpha = i_{11}((g_{11} + g_{12} + i_{14})(z_9 + y_9) - 1) = 0.$$

Since  $\alpha \equiv 3 \pmod J$  and  $3 \in R^*$ , we have  $i_{11} = 0$ . From the position (14, 14) of the same condition we obtain  $i_{14}(i_{14} - 1) = 0$ , therefore,  $i_{14} = 1$ .

Now we will use the condition

$$Con8 := (w_2 x_2 w_2^{-1} x_1 = x_1 w_2 x_2 w_2^{-1}).$$

Its position (14, 2) gives  $s_{13} = 0$ ,

Let us make the block-diagonal basis change that is identical on the submodule, generated by all  $\{V_1, \dots, V_l\}$ , and all  $\{v_i, v_{-i}\}$ , where  $\alpha_i$  is a long root, and with the matrix

$$\begin{pmatrix} \frac{a_{13}}{a_{13}^2 - b_{13}^2} & \frac{-b_{13}}{a_{13}^2 - b_{13}^2} \\ \frac{-b_{13}}{a_{13}^2 - b_{13}^2} & \frac{a_{13}}{a_{13}^2 - b_{13}^2} \end{pmatrix}$$

on all submodules, generated by  $\{v_i, v_{-i}\}$ , where  $\alpha_i$  is a short root. This basis change does not move  $w_i$  for all  $i$ , it is equivalent to unit modulo  $J$ . At the same time we have  $a_{13} = -2$ ,  $b_{13} = 0$ .



The position (13, 13) of

$$Con9 := (h_{\alpha_2}(-1)x_1h_{\alpha_2}(-1)x_1 = 1)$$

gives

$$2h_1 - 2g_{13}h_{11} + h_{13}^2 - 1 = 0,$$

and the position (13, 13) of *Con7* gives

$$-2h_1 + 2g_{13}h_{11} + h_{13}^2 - h_{13} = 0.$$

These two equalities imply  $h_{13} = 1$ .

Now the positions (2, 12) and (2, 13) of *Con9* give

$$\begin{cases} -b_1a_{11} + b_{11}(-b_2 + g_{12} - g_{11}) = 0, \\ -b_12 + b_{11}g_{13} = 0. \end{cases}$$

This system modulo  $J$  is equivalent to

$$\begin{cases} 0 \cdot \bar{b}_1 - 2 \cdot \bar{b}_{11} = 0, \\ -2 \cdot \bar{b}_1 + 0 \cdot \bar{b}_{11} = 0, \end{cases}$$

therefore  $b_1 = b_{11} = 0$ . The same condition (pos. (2, 2)) directly implies  $b_2 = 1$ .

Let us again make the block-diagonal basis change, that is identical on the submodule, generated by all  $\{V_1, \dots, V_l\}$  and all  $\{v_i, v_{-i}\}$ , where  $\alpha_i$  is a short root, and has the matrix

$$\begin{pmatrix} \frac{l_{13}}{l_{13}^2 - m_{13}^2} & \frac{-m_{13}}{l_{13}^2 - m_{13}^2} \\ \frac{-m_{13}}{l_{13}^2 - m_{13}^2} & \frac{l_{13}}{l_{13}^2 - m_{13}^2} \end{pmatrix}$$

on all submodules, generated by  $\{v_i, v_{-i}\}$ , where  $\alpha_i$  is a long root. This basis change does not move  $w_i$  for all  $i$ , and is equivalent to 1 modulo  $J$ . At the same time we have  $l_{13} = 1$ ,  $m_{13} = 0$ .

Now we suppose that from the isomorphism  $\varphi_2$  after the last two basis changes we come to the isomorphism  $\varphi_3$ .

From the position (13, 2) of *Con9* we obtain  $h_1a_2 = -2h_{11}g_2$ , therefore,  $h_1 = \beta_1h_{11}$ ,  $\beta_1 \equiv 0 \pmod{J}$ . Now from (1, 12) of *Con9*  $-a_1a_{11} + a_{11}g_{12} - a_{11}g_{11} - 2h_{11}$ , consequently,  $a_{11} = \beta_2h_{11}$ ,  $\beta_2 \equiv -1 \pmod{J}$ . Similarly, from (1, 13) of *Con9* it follows  $a_1 = 1 + \beta_3h_{11}$ ,  $\beta_3 \equiv 0 \pmod{J}$ , from (12, 1) of *Con9*  $g_{12} = \beta_4h_{11}$ , it follows  $\beta_4 \equiv 0 \pmod{J}$ , from (12, 12) of *Con9* it follows  $g_{11} = 1 + \beta_5h_{11}$ ,  $\beta_5 \equiv 0 \pmod{J}$ , from (12, 1) of *Con9* it follows  $g_1 = \beta_6h_{11}$ ,  $\beta_6 \equiv 0 \pmod{J}$ . Using the position (2, 1) of

$$Con10 := (x_1w_1x_1w_1^{-1} = w_1),$$

we obtain  $a_2 = -1 + \beta_7h_{11}$ ,  $\beta_7 \equiv 0 \pmod{J}$ . From (1, 2) of *Con9* it follows  $h_2 = 1 + \beta_8h_{11}$ ,  $\beta_8 \equiv 0 \pmod{J}$ . Using position (1, 2) of

$$Con11 := (w_2x_2w_2^{-1}x_1 = x_1w_2x_2w_2^{-1}),$$

we obtain  $j_2 = \beta_9h_{11}$ ,  $\beta_9 \equiv 0 \pmod{J}$ . From (13, 12) of *Con11*  $j_{11} = \beta_{10}h_{11}$ ,  $\beta_{10} \equiv 0 \pmod{J}$ , from (1, 11) of *Con11*  $j_{12} = \beta_{11}h_{11}$ ,  $\beta_{11} \equiv 0 \pmod{J}$ , from (1, 1) of *Con7*  $u_1 = \beta_{12}h_{11}$ ,  $\beta_{12} \equiv 0 \pmod{J}$ , from (1, 12) of *Con7*  $u_{12} = \beta_{13}h_{11}$ ,  $\beta_{13} \equiv 3/2 \pmod{J}$ , from (13, 2) of *Con7*  $u_2 = 1/2 + \beta_{13}h_{11}$ ,  $\beta_{13} \equiv 0 \pmod{J}$ .

Using the position (7, 7) of

$$Con12 := (d_2\varphi_2(x_{2\alpha_1+\alpha_2}(1)) = \varphi_2(x_{2\alpha_1+\alpha_2}(1))d_2 = d_2w_1w_2x_1w_2^{-1}w_1^{-1} - w_1w_2x_1w_2^{-1}w_1^{-1}d_2)$$

we obtain  $x_8 = \beta_{14}h_{11}$ ,  $\beta_{14} \equiv 0 \pmod{J}$ , from the position (14, 8) of *Con12* it follows  $x_7 = 1 + \beta_{15}h_{11}$ ,  $\beta_{15} \equiv 0 \pmod{J}$ , from (1, 11) of *Con7* it follows  $u_{11} = \beta_{15}h_{11}$ ,  $\beta_{15} \equiv 0 \pmod{J}$ .

Using the position (1, 12) of

$$\text{Con13} := (w_2 d_2 w_2^{-1} d_2 = 1),$$

we obtain  $t_{12} = \beta_{16} h_{11}$ ,  $\beta_{16} \equiv 0 \pmod{J}$ , from (1, 11) of *Con13* it follows  $t_{11} = \beta_{17} h_{11}$ ,  $\beta_{17} \equiv -3/2 \pmod{J}$ , from (1, 5) of *Con13* it follows  $t_2 = \beta_{18} h_{11}$ ,  $\beta_{18} \equiv 0 \pmod{J}$ , from (2, 2) of *Con13* it follows  $t_1 = 2 + \beta_{19} h_{11}$ ,  $\beta_{19} \equiv 0 \pmod{J}$ , from (7, 4) of *Con12* it follows  $x_4 = \beta_{20} h_{11}$ ,  $\beta_{20} \equiv 0 \pmod{J}$ , from (7, 3) of *Con12* it follows  $x_3 = \beta_{21} h_{11}$ ,  $\beta_{21} \equiv 0 \pmod{J}$ .

Position (13, 14) of

$$\text{Con14} := (x_2 \varphi_2(x_{2\alpha_1+\alpha_2}(1)) = \varphi_2(x_{2\alpha_1+\alpha_2}(1))x_2)$$

gives  $n_{14} = \beta_{22} h_{11}$ ,  $\beta_{22} \equiv -2 \pmod{J}$ , from (13, 3) of *Con14* it follows  $n_3 = \beta_{23} h_{11}$ ,  $\beta_{23} \equiv 0 \pmod{J}$ , from (13, 4) of *Con14* it follows  $n_4 = \beta_{24} h_{11}$ ,  $\beta_{24} \equiv 0 \pmod{J}$ , from (8, 13) of *Con14* it follows  $n_8 = \beta_{25} h_{11}$ ,  $\beta_{25} \equiv 0 \pmod{J}$ .

Finally, from (7, 4) of *Con14* it follows  $\beta_{26} h_{11} = 0$ , where  $\beta_{26} \in R^*$ . Consequently,  $h_{11} = 0$ .

Now position (3, 14) of *Con11* gives  $c_4 = 0$ , position (4, 14) of the same condition gives  $d_4 = 1$ , position (13, 2) gives  $s_{14} = j_1$ .

Position (3, 13) of

$$\text{Con15} := (h_{\alpha_1}(-1)x_2 h_{\alpha_1}(-1)x_2 = 1)$$

gives  $l_3 = j_1$ , position (4, 3) gives  $m_3(m_4 + 1) = 0 \Rightarrow m_3 = 0$ , position (3, 3) gives  $s_3 = 0$ , (4, 4) gives  $m_4 = 1$ , (4, 7) gives  $m_7(1 + n_7) = 0 \Rightarrow m_7 = 0$ , (3, 13) gives  $j_1 = 1$ , (3, 4) gives  $s_4 = -l_4$ .

Position (5, 3) of *Con11* gives  $e_4 l_4 = 0 \Rightarrow e_4 = 0$ , position (4, 13) of the same condition gives  $(d_7 + d_8)n_{13} = 0 \Rightarrow n_{13} = 0$ , position (8, 8) of *Con15* gives  $n_7 = 1$ , position (7, 13) of *Con11* gives  $f_9 = 0$ , position (8, 13) gives  $e_9 = 0$ , position (13, 13) of *Con14* gives  $s_8 = -s_7$ , position (14, 13) gives  $-l_4(-2g_{13} - g_{14}) = s_7$ , and position (3, 13) gives  $l_4(-2g_{13} - g_{14}) = l_7$ , therefore  $s_7 = -l_7$ . Position (13, 5) of *Con11* gives  $j_6 = 0$ , (13, 6) gives  $j_5 = 0$ , (13, 9) gives  $j_{10} = 0$ , (13, 10) gives  $j_9 = 0$ .

Positions (10, 13) and (10, 14) of

$$\begin{aligned} \text{Con16} &:= (\varphi_3(x_{3\alpha_1+\alpha_2}(1)x_{\alpha_2}(1)x_{3\alpha_1+2\alpha_2}(1)) = \varphi_3(x_{\alpha_2}(1)x_{3\alpha_1+\alpha_2}(1))) := \\ &= (w_1 x_2 w_1^{-1} x_2 w_2 w_1 x_2 w_2^{-1} w_2^{-1} = w_1 x_2 w_1^{-1} x_2) \end{aligned}$$

give  $q_9 = q_{11} = 0$ , (7, 13) —  $k_{12} = 0$ , (9, 13) —  $p_9 = 1$ , (2, 13) —  $k_9 = 0$ , (2, 14) —  $k_{11} = 0$ , (5, 14) —  $k_{10} = 0$ , (14, 12) —  $q_{12} = 1$ , (9, 14) —  $p_{11} = 0$ , (4, 14) —  $p_{10} q_{12} = 0 \Rightarrow p_{10} = 0$ , (12, 14) —  $p_{12} = 0$ , (11, 14) —  $q_{10} = 1$ , (3, 10) —  $l_4 = -1$ , (5, 1) —  $k_6(k_2 - 1) = 0 \Rightarrow k_2 = 1$ , (5, 2) —  $k_6 k_1 = 0 \Rightarrow k_1 = 0$ , (4, 1) —  $p_5 = p_6 = 0$ , (13, 5) —  $q_5 = 0$ , (4, 6) —  $q_1 k_6 = 0 \Rightarrow q_1 = 0$ , (1, 6) —  $k_5 k_6 = 0 \Rightarrow k_5 = 0$ , (5, 8) —  $k_6(k_6 - 1) = 0 \Rightarrow k_6 = 1$ , (12, 8) —  $p_2 = 0$ , (13, 6) —  $q_6 = 0$ .

From *Con11* it now follows that  $e_3 = 3$  (pos. (1, 3)),  $f_4 = 0$  (pos. (6, 3)),  $d_9 = 0$  (pos. (9, 3)),  $c_9 = 0$  (pos. (10, 3)),  $e_6 = 0$  (pos. (1, 6)),  $e_{10} = 0$  (pos. (1, 10)),  $e_5 = 1$  (pos. (1, 5)).

From *Con14* it follows  $f_7 = 0$  (pos. (1, 1)),  $f_3 = 0$  (pos. (1, 12)). Position (1, 3) of *Con7* gives  $w_3 = 0$ , positions (3, 4) and (4, 3) of *Con9* give  $v_4 = 0$ .

From (12, 2) of *Con9* we obtain  $g_{13} = 2g_2$ , from (8, 3) we obtain  $e_8 = -c_{10}$ , from (12, 14) of *Con10* we obtain  $g_{14} = -3g_2$ , from (6, 9) we have  $f_{10}(d_3 + f_5) = 0 \Rightarrow f_5 = -d_3$ , from (10, 6) of *Con15* —  $q_2 = -p_1$ , from (1, 7) of *Con11* we have  $e_7 = 3l_7$ , from (1, 8) we obtain  $c_{10} = 3l_7$ , from (12, 3) we have  $g_2 = l_7$ , from (11, 3) it follows  $p_1 = -l_7$ .

Now from

$$\text{Con17} := (d_2 x_2^4 = x_2 d_2)$$

$v_4 = 16v_3$  (pos. (3, 4)),  $v_3 = 1/4$  (pos. (3, 13)),  $w_7 = -3l_7$  (pos. (3, 8)),  $y_{10} = 0$  (pos. (9, 12)),  $y_9 = 4z_{10}$  (pos. (10, 12)),  $z_{10} = x_{10}$  (pos. (11, 9)). From (11, 12) and (12, 12) of *Con6* we have  $z_9 = 0$ ,  $z_{10} = 1/2$ ,

from (9,5) and (9,6)  $z_6 = -y_5$ ,  $y_6 = -4z_5$ , from (4,8)  $v_7 = -3/4l_7$ . Again from *Con17* it follows  $z_5 = 0$  (pos. (11,6)),  $y_5 = -3/2l_7$  (pos. (10,2)). From (3,2) of *Con11* we obtain  $c_6 = -3l_7^2$ , from (4,3) we have  $c_3 = 1 + l_7f_{10}$ , from (4,2) we have  $d_6 = -1 - d_{10}l_7$ , from (6,2) we have  $f_6 = 1 - l_7f_{10}$ , from (4,8) we have  $c_8 = l_7d_3$ , from (12,6)  $c_7 = l_7f_{10}$ , from (9,12)  $d_3 = -l_7f_{10}$ . From (5,7) of *Con7* we obtain  $3l_7(l_7f_{10} - f_{10}) = 0$ . Since  $f_{10} \in R^*$ , we have  $l_7 = 0$ . From *Con11* it follows  $c_5 = 0$ , from (4,7) of *Con9* it follows  $d_7 = 0$ , from (4,3) it follows  $d_5 = 0$ , from (3,9) of *Con10* it follows  $d_{10} = 1$ , from (5,9) we have  $f_{10} = -3$ , from (3,7) we have  $d_8 = 1$ , from (3,5) we have  $f_8 = -2$ .

Now, finally, we see that  $x_1 = x_{\alpha_1}(1)$ ,  $x_2 = x_{\alpha_2}(1)$ ,  $d_2 = h_{\alpha_2}(2)$ . It is easy to check that  $\varphi_3(h_{\alpha_1}(t)) = h_{\alpha_1}(s)$  and  $\varphi_3(h_{\alpha_2}(t)) = h_{\alpha_2}(s)$  for some  $s \in R^*$ . Since all roots of the same length are conjugate up to the action of  $W$ , then  $\varphi_3(x_{\alpha_i}(1)) = x_{\alpha_i}(1)$  and  $\varphi_3(h_{\alpha_i}(t)) = h_{\alpha_i}(s)$  for some  $s \in R^*$ .

## 6. PROOF OF THEOREM 1.

Now we have stated that for both root systems under consideration  $\varphi_3(x_{\alpha_i}(1)) = x_{\alpha_i}(1)$ ,  $\varphi_3(h_{\alpha_i}(t)) = h_{\alpha_i}(s)$ ,  $i = \pm 1, \dots, \pm m$ ,  $t, s \in R^*$ .

For any long root  $\alpha_j$  there exists a root  $\alpha_k$  such that  $h_{\alpha_k}(t)x_{\alpha_j}(1)h_{\alpha_k}(t)^{-1} = x_{\alpha_j}(t)$ . Therefore,  $\varphi_3(x_{\alpha_j}(t)) = x_{\alpha_j}(s)$ . From these conditions and commutator conditions it follows that  $\varphi_3(x_{\alpha_j}(t)) = x_{\alpha_j}(s)$  for all  $\alpha_j \in \Phi$ .

Let us denote the mapping  $t \mapsto s$  by  $\rho : R^* \rightarrow R^*$ . If  $t \notin R^*$ , then  $t \in J$ , i.e.,  $t = 1 + t_1$ , where  $t_1 \in R^*$ . Then  $\varphi_3(x_{\alpha}(t)) = \varphi_3(x_{\alpha}(1)x_{\alpha}(t_1)) = x_{\alpha}(1)x_{\alpha}(\rho(t_1)) = x_{\alpha}(1 + \rho(t_1))$ ,  $\alpha \in \Phi$ . Therefore, if we extend the mapping  $\rho$  to the whole ring  $R$  (with the formula  $\rho(t) := 1 + \rho(t - 1)$  for  $t \in J$ ), then we obtain  $\varphi_3(x_{\alpha}(t)) = x_{\alpha}(\rho(t))$  for all  $t \in R$ ,  $\alpha \in \Phi$ . It is clear that  $\rho$  is injective, additive, multiplicative on invertible elements. Since every element of the ring  $R$  is a sum of two invertible elements, we have that  $\rho$  is also multiplicative on uninvertible elements of the ring, i.e., is an isomorphism from  $R$  to some its subring  $R'$ . Note that in this situation  $CE(\Phi, R)C^{-1} = E(\Phi, R')$  for some matrix  $C \in GL(V)$ . Let us show that  $R' = R$ .

**Lemma 2.** *Elementary Chevalley group  $E_{ad}(R, \Phi)$  generates  $M_n(R)$  as a ring.*

*Proof.* Let us consider the case of the root system  $B_2$ , since the case  $G_2$  is completely similar. The matrix  $(x_{e_1+e_2}(1) - 1)^2$  is  $-2E_{5,6}$  ( $E_{ij}$  is a matrix unit). Multiplying it to some appropriate diagonal matrix, we obtain an arbitrary matrix of the form  $\alpha \cdot e_{12}$  (since  $-2 \in R^*$  and invertible elements of  $R$  generate  $R$ ). Then  $w_{e_1}\alpha E_{5,6} = \alpha E_{8,5}$ ,  $w_{e_1}\alpha E_{5,6}w_{e_1} = \alpha E_{8,7}$ ,  $w_{e_2}\alpha E_{5,6} = \alpha E_{7,6}$ ,  $w_{e_2}\alpha E_{5,6}w_{e_1} = \alpha E_{7,7}$ ,  $w_{e_2}\alpha E_{5,6}w_{e_2} = \alpha E_{7,8}$ ,  $w_{e_1}\alpha E_{5,6}w_{e_2} = \alpha E_{8,8}$ ,  $w_{e_1+e_2}\alpha E_{5,6} = \alpha E_{6,6}$ . These matrices generate a subring of the matrix ring, generated by  $E_{i,j}$ ,  $4 \leq i, j \leq 8$ . Similarly, with the help of  $(x_{e_1}(1) - 1)^2$  we can obtain a subring, generated by  $E_{i,j}$ ,  $1 \leq i, j \leq 4$ . With these matrix units and elements  $x_{\alpha}(1)$  we can generate the subring  $M_8(R)$ . Now let us subtract from  $x_{e_2}(1) - 1$  appropriate matrix units, and we obtain the matrix  $E_{10,4} - 2E_{3,9} + E_{3,10}$ . Multiplying it (from the right side) to  $E_{4,i}$ ,  $1 \leq i \leq 8$ , we obtain all  $E_{10,i}$ ,  $1 \leq i \leq 2m$ . With the help of the Weil group we have all  $E_{i,j}$ ,  $8 < i \leq 9$ ,  $1 \leq j \leq 8$ . Now we have the matrix  $-2E_{3,9} + E_{3,10}$ . Multiplying it (from the left side) to  $E_{2m+1,1}$ , we have  $E_{2m+1,2m+1}$ . With the help of last two matrices we have  $E_{3,9}$ , therefore  $E_{i,j}$ ,  $1 \leq i \leq 8$ ,  $8 < j \leq 9$ . Thus we obtain all matrix units, and so the whole matrix ring  $M_n(R)$ .  $\square$

**Lemma 3.** *If for some  $C \in GL(V)$  we have  $CE(\Phi, R)C^{-1} = E(R', \Phi)$ , where  $R'$  is a subring in  $R$ , then  $R' = R$ .*

*Proof.* Suppose that  $R'$  is a proper subring in  $R$ .

Then  $CM_n(R)C^{-1} = M_n(R')$ , since the group  $E(\Phi, R)$  generates the ring  $M_n(R)$ , and the group  $E(\Phi, R') = CE(\Phi, R)C^{-1}$  generates the ring  $M_n(R')$ . It is impossible, since  $C \in GL_n(R)$ .  $\square$

Consequently, we have proved that  $\rho$  is an automorphism of the ring  $R$ . So the composition of the initial isomorphism  $\varphi'$  and some basis change with a matrix  $C \in GL_n(R)$  (that maps  $E(\Phi, R)$  onto itself), is a ring automorphism  $\rho$ . Thus  $\varphi'' = i_{C^{-1}} \circ \rho$ .

Therefore, Theorem 1 is proved.

## 7. PROOF OF THEOREM 2.

**Proof of Theorem 2.** Suppose that we have a Chevalley group  $G_{ad}(\Phi, R)$  and its automorphism  $\varphi$ . Since the elementary subgroup  $E_{ad}(\Phi, R)$  is the commutant of  $G_{ad}(\Phi, R)$ , then it is mapped onto itself under the action of  $\varphi$ . On the elementary subgroup  $\varphi$  is a composition of standard automorphisms:  $\varphi = i_C \circ \rho$ . Consider a mapping  $\varphi' = \rho^{-1} \circ i_C^{-1} \circ \varphi$ . It is an isomorphism from the Chevalley group  $G_{ad}(\Phi, R)$  onto some subgroup  $G \subset GL_n(R)$ , that it identical on the elementary subgroup. We know that  $G_{ad}(\Phi, R) = E_{ad}(\Phi, R)T_{ad}(\Phi, R)$ ,  $T_{ad}(\Phi, R)$  consists of such elements  $h_\chi$ , that  $\chi : \Phi \rightarrow R^*$  is a homomorphism,  $h_\chi x_\alpha(t) h_\chi^{-1} = x_\alpha(\chi(\alpha) \cdot t)$ . Every  $h_\chi$  commutes with all  $h_\alpha(t)$ ,  $t \in R^*$ .

Consider a matrix  $A = \varphi'(h_\chi)$ . Since  $A$  commutes with all  $h_\alpha(t)$ ,  $\alpha \in \Phi$ ,  $t \in R^*$ , it has the form

$$A = \begin{pmatrix} D & 0 \\ 0 & C \end{pmatrix},$$

where  $D$  is a diagonal matrix  $(2m) \times (2m)$ ,  $C$  is some matrix  $l \times l$ . For all  $\alpha \in \Phi$  and  $t \in R$  we have the condition

$$Ax_\alpha(t)A^{-1} = x_\alpha(\chi(\alpha) \cdot t).$$

By direct calculations (as it was done in the previous sections) we see that this condition implies  $A = a_\chi h_\chi$ , where  $a_\chi \in R^*$ . Let  $h_{t_1, \dots, t_l}$  be a homomorphism from the root lattice to  $R^*$  such that every simple root  $\alpha_i$  is mapped to  $t_i$ . Then from  $(h_{t_1, \dots, t_l})^2 \cdot (h_{1, t^{-1}, 1, \dots, 1}) = h_{\alpha_1}(t)$  it follows that for all  $\chi$   $a_\chi^3 = 1$ . Since  $\varphi'(h_\chi) \in SL_n(R)$ , we have  $a_\chi^n = 1$ . So (since 10, 14, that are dimensions of adjoint representations of Chevalley groups  $B_2$  and  $G_2$ , do not divided to 3), we have  $a_\chi = 1$  for all  $\chi$ , therefore,  $\varphi'$  is identical. Theorem is proved.

**Corollary 1.** *Any automorphism of an (elementary) adjoint Chevalley group of types  $B_2$  (or  $G_2$ ) over local commutative ring with  $1/2$  ( $1/6$ ) induces an automorphism of the matrix ring  $M_n(R)$ , where  $n$  is a dimension of adjoint representation.*

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